problem started at end of last lecture:

saw previously: $\vec{x}(t) = \underbrace{\phi(t)\,\phi^{-1}(0)}_{e^{At}}\,\underbrace{\vec{x}(0)}_{\vec{x}_0}$

can find e^{At} by solving $\vec{x}\,' = A\vec{x}$

revisit $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ has $\phi(t) = \begin{bmatrix} 2e^{5t} & e^{-2t} \\ e^{5t} & -3e^{-2t} \end{bmatrix}$

also $\phi^{-1}(0) = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$ ← extraneous negatives canceled out

then $e^{At} = \frac{1}{7}\begin{bmatrix} 2e^{5t} & e^{-2t} \\ e^{5t} & -3e^{-2t} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$

$= \frac{1}{7}\begin{bmatrix} 6e^{5t} + e^{-2t} & 2e^{5t} - 2e^{-2t} \\ 3e^{5t} - 3e^{-2t} & e^{5t} + 6e^{-2t} \end{bmatrix}$

ex. use exponential matrix to solve IVP

$\vec{x}\,' = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}\vec{x}$ given init. cond. $\vec{x}(0) = \begin{bmatrix} 19 \\ 29 \\ 39 \end{bmatrix}$

same A as prev.

char eqn. $(2-\lambda)\det\begin{bmatrix} 2-\lambda & 6 \\ 0 & 2-\lambda \end{bmatrix} = 0$

$(2-\lambda)(2-\lambda)(2-\lambda) = 0$

$\lambda = 2 \quad k = 3$

then EV eq'n : $(A - 2I_3)\vec{v} = \vec{0}$ becomes

$\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{II}} 6v_3 = 0 \Rightarrow v_3 = 0$

I : $3v_2 = 0 \Rightarrow v_2 = 0$

$\therefore v_1$ is a free variable

let $v_1 = 1$

$EV_{\lambda=2}: \begin{bmatrix}1\\0\\0\end{bmatrix}$ no other linearly indep. sol'ns

however, since know that $A = \begin{bmatrix}2&3&4\\0&2&6\\0&0&2\end{bmatrix}$ has a

corresponding $e^{At} = \begin{bmatrix}e^{2t} & 3te^{2t} & (4t+9t^2)e^{2t}\\ 0 & e^{2t} & 6te^{2t}\\ 0 & 0 & e^{2t}\end{bmatrix}$ ← see notes from last time

use thm saw previously $\vec{x}(t) = e^{At}\vec{x}(0)$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix}19\\29\\39\end{bmatrix}$$

$$= \begin{bmatrix}(19+243t+351t^2)\,e^{2t}\\ (29+234t)\,e^{2t}\\ 39\,e^{2t}\end{bmatrix}$$

knowing fundamental matrix (ϕ) and its inverse matrix (ϕ^{-1}) is advantageous in that can easily change init. conditions and re-compute

Section 7.1 Laplace Transform Methods

Laplace transformation $\mathcal{L}$ involves the operation of integration and generates a new function $\mathcal{L}\{f(t)\} = F(s)$

antideriv. new variable

Laplace transform converts a DE containing $f(t)$ into an algebraic eqn, $F(s)$.

def'n: given $f(t)$ defined on $t \geq 0$

the Laplace transform of f is F:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

for all s for which improper integral converges

ex. given $f(t) \equiv 1$ for $t \geq 0$

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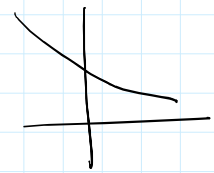
$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} (1)\, dt$$
$$= \lim_{b\to\infty} \int_0^b e^{-st}\, dt$$
$$= -\frac{1}{s} \lim_{b\to\infty} e^{-st} \Big|_0^b$$
$$= -\frac{1}{s}\left(\lim_{b\to\infty} \cancelto{0}{e^{-sb}} - 1\right)$$



$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}} \quad \text{for } s > 0$$

ex. given $f(t) = e^{at}$ for $t \geq 0$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$
$$= \int_0^\infty e^{-(s-a)t}\, dt$$
$$= \frac{1}{-(s-a)}\left[\lim_{t\to\infty} \cancelto{0}{e^{-(s-a)t}} - 1\right]$$
$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}} \quad s > a$$

special case:

$$\mathcal{L}\{t^a\} = \underbrace{\int_0^\infty e^{-t} t^{x-1}\, dt}_{\text{called gamma function}: \ \Gamma(x)} \quad x > 0$$

then $\Gamma(1) = \int_0^\infty e^{-t} (1)\, dt$
$$= -\left(\lim_{t\to\infty} e^{-t} - 1\right) = 1 \qquad \boxed{\Gamma(1) = 1}$$

also $\Gamma(x+1) = \int_0^\infty e^{-t} t^x\, dt = x\Gamma(x) \quad x > 0$

$$\boxed{\Gamma(x+1) = x\Gamma(x)}$$

let n be $\mathbb{Z}^+$

follows that $\Gamma(n+1) = n\Gamma(n)$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$

$7! = 7 \cdot 6!$

$$= n(n-1)(n-2)\ldots 2\,\Gamma(2)$$

$$= n(n-1)(n-2)\ldots 2\cdot 1\cdot \underbrace{\Gamma(1)}_{1}$$

$$\boxed{\Gamma(n+1) = n!}$$

solving $\mathcal{L}\{t^a\}$ where $a \in \mathbb{R}$ and $a > -1$

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a \, dt$$

let $u = st \Rightarrow t = \frac{u}{s}$

$dt = \frac{du}{s}$

$$= \int_0^\infty e^{-u} \frac{u^a}{s^a} \frac{du}{s}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a \, du$$

$\leftarrow$ gamma function format where $a = x-1$ $\therefore x = a+1$

$$= \frac{1}{s^{a+1}} \Gamma(a+1)$$

$$= \frac{1}{s^{a+1}} a!$$

for all $s>0$

apply to $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $s>0$

$n=1$ $\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}}$

$n=2$ $\boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3}}$

$n=3$ $\boxed{\mathcal{L}\{t^3\} = \frac{6}{s^4}}$

thm: given a,b are constants:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

spl case $\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$

explore $\mathcal{L}\{t^{n/2}\}$ format

ex. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$= \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$

$= \boxed{\frac{3\sqrt{\pi}}{4}}$

ex. $\mathcal{L}\{3t^2 + 4t^{3/2}\}$

$= 3\mathcal{L}\{t^2\} + 4\mathcal{L}\{t^{3/2}\}$

$= 3\cdot\frac{2}{s^3} + 4\frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}$

$= \frac{6}{s^3} + \frac{\cancel{4}\cdot 3\sqrt{\pi}}{\cancel{4}\, s^{5/2}}$ $\longrightarrow \sqrt{s^5}$

$= \boxed{\frac{6}{s^3} + 3\sqrt{\frac{\pi}{s^5}}}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{t^{3/2}\} = \frac{\left(\frac{3}{2}\right)!}{s^{5/2}}$ $\quad n! = \Gamma(n+1)$

$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}$

know $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

recall $\cosh(kt) = \frac{1}{2}\left(e^{kt} + e^{-kt}\right)$

if $k>0$ then $\mathcal{L}\{\cosh(kt)\} = \frac{1}{2}\mathcal{L}\{e^{kt}\} + \frac{1}{2}\mathcal{L}\{e^{-kt}\}$

$= \frac{1}{2}\cdot\frac{1}{s-k} + \frac{1}{2}\cdot\frac{1}{s+k}$

$= \frac{1}{2}\left(\frac{s+k+s-k}{s^2-k^2}\right)$

$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}s}{s^2-k^2}$

$\therefore \mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2} \qquad k>0$

similarly, $\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2} \qquad s>k>0$

more importantly, $\boxed{\begin{aligned}\mathcal{L}\{\cos(kt)\} &= \frac{s}{s^2+k^2}\\ \mathcal{L}\{\sin(kt)\} &= \frac{k}{s^2+k^2}\end{aligned}}$

to solve for $f(t)$, use inverse

if $F(s) = \mathcal{L}\{f(t)\}$ then $f(t) = \mathcal{L}^{-1}\{F(s)\}$

ex. know $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ then $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2$

also, let $k=3$ if $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$ then $\Rightarrow \mathcal{L}^{-1}\left\{\frac{2}{s^2+9}\right\} = \frac{1}{3}\cdot 2\sin 3t$

$= \frac{2}{3}\sin 3t$

ex. $\mathcal{L}\{3e^{2t} + 2\sin^2 3t\}$

$= 3\mathcal{L}\{e^{2t}\} + \mathcal{L}\{1 - \cos 6t\}$

$\cos 2(3t) = 1 - 2\sin^2(3t)$

$-(\cos 6t - 1) = + 2\sin^2 3t$

$1 - \cos 6t$

$= 3\cdot\frac{1}{s-2} + \frac{1}{s} - \mathcal{L}\{\cos 6t\}$

$= \boxed{\frac{3}{s-2} + \frac{1}{s} - \frac{s}{s^2+36}}$

w/ LCD $= \frac{3s^3 + 144s - 72}{s(s-2)(s^2+36)}$